ISRAEL JOURNAL OF MATHEMATICS, Vol. 34, No. 4, 1979

FINITARY ISOMORPHISMS OF IRREDUCIBLE MARKOV SHIFTS

BY

MICHAEL KEANE AND MEIR SMORODINSKY

ABSTRACT

It is shown that irreducible finite state, Markov shifts of the same entropy and period are *finitarily isomorphic*.

1. Introduction

The purpose of this paper is to prove the following result.

THEOREM. *Irreducible finite memory Markov shifts on finite state spaces are finitarily isomorphic iff they have the same period and the same entropy.*

This paper is a continuation of [8] and [9] in which we introduced the marker method of constructing isomorphisms, different from the method developed by Ornstein. The marker method yields finitary maps, but is more limited in scope. Using Ornstein's method, the (non-finitary) isomorphism theorem for Markov shifts was proved in [6] and [4].

More recently, a finitary homomorphism theorem for Markov shifts of different entropies was proved by Akcoglu, del Junco and Rahe in [1]. Also, Adler and Marcus [3] gave a topological equivalence theorem for shifts of finite type. In particular, it follows from this that in the special class of Markov shifts of maximal entropy, two are isomorphic if they have the same entropy and period.

One interesting consequence of our theorem above is obtained by considering a hyperbolic automorphism of a torus. Since these have nice Markov partitions, they are finitarily isomorphic to Markov shifts and hence by our theorem to Bernoulli schemes. Thus there is a generating partition into *open* sets (with boundary measure zero) which is measure-theoretically independent under the

Received March 22, 1979

automorphism. That this result is in a certain sense optimal was shown by Bowen [5]; he proves that such a partition cannot have piecewise smooth boundaries.

The question of whether every ergodic automorphism of a torus is finitarily isomorphic to a Bernoulli (or equivalently Markov) shift remains open. Katznelson [7] proved this for non-finitary isomorphism.

2. Preliminaries

In the proof of the theorem, we shall be constructing new stochastic processes in several ways from given ones. In this paragraph we describe the constructions used and their properties.

Let $X = (X_n)_{n \in Z}$ be a stationary ergodic process on a finite state space $A = \{a_1, a_2, \cdots, a_m\}$.

DEFINITION 1. The process $(X_n^{(k)})_{n \in Z}$, called the k-stringing of X, is defined as follows, where k is a fixed positive integer. The state space of $X^{(k)}$ is $A^k = A \times \cdots \times A$ k times, and

$$X_n^{(k)} = X_n X_{n+1} \cdots X_{n+k-1} \qquad (n \in Z).$$

LEMMA 1. *X and $X^{(k)}$ are finitarily (and even continuously) isomorphic. If X is a Markov process, then $X^{(k)}$ is also a Markov process.*

The proof is obvious.

DEFINITION 2. Let $A' \subseteq A$ be a subset of the set of states of X, and let b be a symbol not belonging to A. We say that the process X', defined by

$$X'_n = \begin{cases} X_n & \text{if } X_n \notin A', \\ b & \text{if } X_n \in A', \end{cases}$$

is obtained from X by *collapsing A'*.

DEFINITION 3. Let $b_1, \cdots, b_l$ be symbols not belonging to A, $q_1, \cdots, q_l$ a probability vector, and $a_i \in A$. We say that *the process $\hat{X}$ is obtained from X by independently splitting a_i according to $q_1, \cdots, q_l$* if $\hat{X}$ is defined as follows: The states of $\hat{X}$ are $b_1, \cdots, b_l, a_1, \cdots, a_{i-1}, a_{i+1}, \cdots, a_m$, and if $c_0, \cdots, c_r$ is a sequence of such states with $c_{j_1} = b_{i_1}, \cdots, c_{j_s} = b_{i_s}$, all other c_j's being a's, then

$$\mathrm{P}[\hat{X}_n = c_0, \cdots, \hat{X}_{n+r} = c_r]$$

$$= \left(\prod_{t=1}^{s} q_{i_t}\right) \mathrm{P}[X_{n+j_t} = a_i, 1 \leqq t \leqq s, X_{n+j} = c_j \text{ for the other } j\text{'s}].$$

DEFINITION 4. The state a_i of X is said to be a *renewal state* if for each fixed n, the σ-algebras $\mathfrak{A}(X_{n+1}, X_{n+2}, \cdots)$ and $\mathfrak{A}(\cdots, X_{n-2}, X_{n-1})$ are conditionally independent given the event $[X_n = a_i]$.

Note that X is a Markov process iff each state of A is a renewal state. The proofs of the following lemmas are obvious.

LEMMA 2. *Let $A' \subseteq A$, $a_i \not\in A'$ such that a_i is a renewal state for X. Then a_i is a renewal state for X', obtained from X by collapsing A'.*

LEMMA 3. *If X and $\hat{X}$ are as in Definition 3, with entropies h and $\hat{h}$ respectively, then*

$$\hat{h} = h + \mathrm{P}[X_0 = a_i] \cdot h(q_1, \cdots, q_l),$$

where

$$h(q_1, \cdots, q_l) = -\sum_{j=1}^{l} q_j \log q_j.$$

DEFINITION 5. Let $a_i \in A$. The *distribution of the state a_i* is defined as the *process X* obtained by setting

$$X_n = \begin{cases} 0 & \text{if } X_n \neq a_i, \\ 1 & \text{if } X_n = a_i. \end{cases}$$

3. Markers for Markov shifts

To prove our theorem, we may obviously assume that the Markov shifts are mixing (= of period one), and in view of the finitary isomorphism theorem for Bernoulli schemes [9], it suffices to show that a given Markov shift is finitarily isomorphic to some Bernoulli scheme, which we may choose at our discretion. The Markov shift may be assumed to be of memory one (Lemma 1).

To do this, we start, with a Mixing Markov shift X of memory one, and construct two processes Y and Z such that for some $k \geqq 1$,

(1) Z is a Bernoulli process (independent process),

(2) $X^{(k)}$ and Y have renewal states with the same distribution,

(3) $Z^{(k)}$ and Y have renewal states with the same distribution,

(4) $h(X) = h(Y) = h(Z)$.

Once we have constructed Y and Z, the methods of [9] yield with minor modifications (which we shall not detail here) finitary isomorphisms between $X^{(k)}$ and Y, and between Y and $Z^{(k)}$, using their respective renewal states with the same distributions as markers. Thus it suffices to find Y, Z and $k \geqq 1$ with the above properties (1)–(4), and the rest of the paper is devoted to that purpose.

LEMMA 4. *Let X be a mixing Markov shift with state space $A = \{a_1, \cdots, a_m\}$, $m \geqq 2$. There exists a state $a_g \in A$ such that for all integers k, there is an allowable sequence*

$$\alpha^0 = \alpha_1^0 \alpha_2^0 \cdots \alpha_k^0 \in A^k$$

with $\alpha_i^0 \neq \alpha_g$ for all $1 \leqq i \leqq k$.

REMARK. By *allowable sequence* we mean a sequence α^0 for which $\mathrm{P}[X_1 = \alpha_1^0, \cdots, X_k = \alpha_k^0]$ is strictly positive.

PROOF. Since $m \geqq 2$ and X is mixing, there exists a state, say a_h, of A which leads to at least two different states (one of which may be a_h itself). Let $a_h a_f \cdots a_h$ denote the shortest allowable sequence from a_h to a_h (there is at least one since X is mixing), and let $a_g \neq a_f$ such that a_h leads to a_g also. Then a_g cannot appear in $a_h \dot{a}_f \cdots a_h$, because otherwise this sequence could be shortened by placing a_h immediately before the first occurrence of a_g and removing the initial elements. (Note that if a_h leads to a_h, then $a_h a_h$ is the shortest sequence.) Therefore in the infinite sequence

$$a_h a_f \cdots a_h a_f \cdots a_h a_f \cdots$$

made up by concatenating the shortest sequence with itself, any finite sequence is allowable and does not contain a_g.

DEFINITION 6. Let a_g be as in Lemma 4. Choose an integer k_0 such that for all $k \geqq k_0$, and for all $1 \leqq i \leqq m$, there is an allowable sequence

$$\alpha^i = \alpha_1^i \alpha_2^i \cdots \alpha_k^i$$

with $\alpha_1^i = a_g$ and $\alpha_k^i = a_i$. This is possible because X is mixing.

In the following, k will be chosen later, but we assume that k is fixed and the sequences $\alpha^0, \alpha^1, \cdots, \alpha^m$ are chosen as above.

Now let $W = (W_n)_{n \in Z}$ be a Bernoulli process with states $b_0, b_1, \cdots, b_m$ and probabilities $q_0, q_1, \cdots, q_m$. Form the process $X \times W = \tilde{Y}$ and let $\tilde{Y}^{(k)}$ be the k-stringing of $\tilde{Y}$. We partition the states of $\tilde{Y}^{(k)}$ into three disjoint subsets M, N, and O, defined as follows:

$$M = \bigcup \Big\{ \alpha^i \times \underbrace{\Big(b_0, b_0, \cdots, b_0, b_i \Big)}_{\text{length } k} : 1 \leqq i \leqq m \Big\},$$

$$N = \{\alpha^0 \times (b_{i_1}, \cdots, b_{i_k}) : (b_{i_1}, \cdots, b_{i_k}) \in \{b_0, \cdots, b_m\}^k\},$$

$$O = \text{all other states of } \tilde{Y}^{(k)}.$$

Let Y' denote the process obtained from $\tilde{Y}^{(k)}$ by collapsing M, N and O (separately). Thus Y' has three states which we shall of course denote also by M, N, and O. Since the probabilities of M and N tend to zero as $k \to \infty$ (regardless of the choice of W) we can choose k so large that $h(Y') < h(X)$.

Now we use the very pretty idea of Akcoglu, del Junco and Rahe in [1]. Their Lemma 6.2 says that we may choose the probabilities $q_0, q_1, \cdots, q_m$ such that the distribution of M in Y' is the same as the distribution of some state of a k-stringing of a well-chosen Bernoulli process Z, which we may take to have the same entropy as X.

Finally we can, in view of Lemma 5, split the state 0 independently in such a fashion to obtain a process Y, with states $M, N, O_1, O_2, \cdots, O_l$, such that $h(Y) = h(X)$. Since the state N of Y obviously has the same distribution as the state α^0 of $X^{(k)}$, and since the state M of Y has the same distribution as a state of $Z^{(k)}$, Z chosen as indicated above, and since $h(X) = h(Y) = h(Z)$, we only need to verify the following lemma to complete our proof.

LEMMA 5. *M and N are renewal states for Y' (and hence also for Y, since independent splitting of O does not destroy the renewal property).*

PROOF. Let E' be an event depending on $Y'_1, Y'_2, \cdots$, and F' an event depending on $Y'_{-1}, Y'_{-2}, \cdots$. To show that M and N are renewal states, we must show that E' and F' are conditionally independent under $Y'_0 = M$ and $Y'_0 = N$. E' is $((X_1, W_1), (X_2, W_2), \cdots)$-measurable and F' is $(\cdots (X_{-1}, W_{-1}), (X_0, W_0), \cdots, (X_{k+1}, W_{k-1}))$-measurable. The event $[Y'_0 = N]$ forces $Y_n = N$ or O for $0 \leqq n \leqq k-1$ since $Y_n = M$ is excluded because α^0 does not contain a_g. Therefore, E' becomes $((X_k W_k), (X_{k+1}, W_{k+1}) \cdots)$-measurable and the relative independence of E' and F' follows from the Markov property of $X \times W$.

Similarly, $[Y'_0 = M]$ forces $Y'_n = O$, $-k+1 \leqq n < 0$ since N is excluded as above and M is excluded because of the sequence $(b_0, \cdots, b_0, b_i)$, $i \neq 0$.

So, F' becomes $(\cdots (X_2 W_{-2})(X_{-1} W_{-1}))$-measurable and therefore relatively independent of E'. □

REFERENCES

1. M. A. Ackoglu, A. del Junco and M. Rahe, *Finitary codes between Markov processes*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **47** (1979), 305–314.

2. R. L. Adler and B. Weiss, *Similarity of automorphisms of the torus*, Mem. Amer. Math. Soc. **98** (1970),

3. R. L. Adler and B. Marcus, *Topological entropy and equivalence of dynamical systems*, Mem. Amer. Math. Soc. **219.**

4. R. L. Adler, P. Shields and M. Smorodinsky, *Irreducible Markov shifts*, Ann. Math. Statist. **43** (1972), 1027–1029.

5. R. Bowen, *Smooth partitions of Anosov diffeomorphisms are weak Bernoulli*, Israel J. Math. **21** (1975), 95–100.

6. N. Friedman and D. S. Ornstein, *An isomorphism of weak Bernoulli transformations*, Advances in Math. **5** (1971), 365–394.

7. Y. Katznelson, *Ergodic automorphisms of T^n are Bernoulli shifts*, Israel J. Math. **10** (1971), 186–195.

8. M. Keane and M. Smorodinsky, A class of finitary codes, Israel J. Math. **26** (1977), 352–371.

9. M. Keane and M. Smorodinsky, *Bernoulli schemes of the same entropy are finitarily isomorphic*, Ann. of Math. **109** (1979), 397–406.

10. M. Keane and M. Smorodinsky, *The finitary isomorphism theorem for Markov shifts*, Bull. Amer. Math. Soc. (new series) **1** (1979), 436–438.

DEPARTMENT DE MATHÉMATIQUE ET INFORMATIQUE
UNIVERSITÉ DE RENNES
RENNES, FRANCE

AND

DEPARTMENT OF STATISTICS
TEL AVIV UNIVERSITY
TEL AVIV, ISRAEL